(19.1) Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) \( f(x) = x^{17} \sin x - e^x \cos 3x \) on \([0, \pi]\).

\textit{Solution.} Since \( f \) is continuous on \([0, \pi]\), it follows from Theorem 19.2 in the textbook that it is also uniformly continuous on \([0, \pi]\).

(b) \( f(x) = x^3 \) on \([0, 1]\).

\textit{Solution.} Since \( f \) is continuous on \([0, 1]\), it follows from Theorem 19.2 in the textbook that it is also uniformly continuous on \([0, 1]\).

(c) \( f(x) = x^3 \) on \((0, 1]\).

\textit{Solution.} Since \( f \) can be extended to a continuous function \( \tilde{f} \) with \( \tilde{f}(x) = x^3 \) and \( \text{dom}(\tilde{f}) = [0, 1] \), it follows that, by Theorem 19.5 in the textbook, \( f \) is uniformly continuous.

(d) \( f(x) = x^3 \) on \( \mathbb{R} \).

\textit{Solution.} The function \( f \) is NOT uniformly continuous on \( \mathbb{R} \). The reason is that we may find two sequences \( (x_n) \) and \( (y_n) \) such that \( x_n - y_n \to 0 \) but there is \( \epsilon_0 > 0 \) such that \( |f(x_n) - f(y_n)| \geq \epsilon_0 \) for every \( n \). For this purpose, we may take \( x_n = n + \frac{1}{n} \) and \( y_n = n \). Note that

\[
\begin{align*}
    f(x_n) - f(y_n) &= (n + \frac{1}{n})^3 - n^3 \\
    &= n^3 + 3n^2 \frac{1}{n} + 3n \left( \frac{1}{n} \right)^2 + \left( \frac{1}{n} \right)^3 - n^3 \\
    &= 3n^2 \frac{1}{n} = 3n \\
    &\geq \frac{3}{2n}.
\end{align*}
\]

We may let \( \epsilon_0 = 3 \).

(e) \( f(x) = \frac{1}{x^2} \) on \((0, 1]\).

\textit{Solution.} The function \( f \) is not uniformly continuous on \((0, 1]\). In fact, let \( s_n = 1/n \) for \( n \in \mathbb{N} \); then \( s_n \) is a Cauchy sequence in \((0, 1]\). Since \( f(s_n) = n^3 \) and \( \lim n^3 = \infty \), \( f(s_n) \) is not a Cauchy sequence. Therefore, by Theorem 19.4, \( f \) is not uniformly continuous on \((0, 1]\).

(f) \( f(x) = \sin \left( \frac{1}{x^2} \right) \) on \((0, 1]\).

\textit{Solution.} The function \( f \) is not uniformly continuous; the argument here is completely analogous to the previous exercise. Let \( s_n = 1/n \) for \( n \in \mathbb{N} \); then \( s_n \) is a Cauchy sequence in \((0, 1]\). Since \( f(s_n) = \sin n^2 \) and \( \lim \sin n^2 \) does not exist, \( f(s_n) \) is not a Cauchy sequence. Therefore, by Theorem 19.4, \( f \) is not uniformly continuous on \((0, 1]\).
(g) \( f(x) = x^2 \sin \left( \frac{1}{x} \right) \) on \((0, 1] \).

**Solution.** The function \( f \) is uniformly continuous on \((0, 1] \), since there is an extension \( \tilde{f} \) of \( f \) defined by

\[
\tilde{f}(x) = \begin{cases} 
  x^2 \sin \left( \frac{1}{x} \right), & \text{if } x \in (0, 1] \\
  0, & \text{if } x = 0
\end{cases}
\]

which is continuous on \([0, 1] \).

(19.2) Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the \( \varepsilon - \delta \) property in Definition 19.1.

(a) \( f(x) = 3x + 11 \) on \( \mathbb{R} \).

**Proof.** Let \( \varepsilon > 0 \), and fix \( \delta = \frac{\varepsilon}{3} \). If \( x, y \in \mathbb{R} \) and \( |x - y| < \delta \), then

\[
|f(x) - f(y)| = |3x + 11 - (3y + 11)| = 3|x - y| < 3\delta = 3 \left( \frac{\varepsilon}{3} \right) = \varepsilon.
\]

Hence \( f \) is uniformly continuous on \( \mathbb{R} \). \( \square \)

(b) \( f(x) = x^2 \) on \([0, 3]\).

**Proof.** Let \( \varepsilon > 0 \), and fix \( \delta = \frac{\varepsilon}{6} \). If \( x, y \in [0, 3] \) and \( |x - y| < \delta \), then \( x + y \leq 6 \) and

\[
|f(x) - f(y)| = |x^2 - y^2| = |(x - y)(x + y)| = |x - y| \cdot |x + y| 
\leq 6|x - y| < 6 \left( \frac{\varepsilon}{6} \right) = \varepsilon.
\]

Therefore, \( f \) is uniformly continuous on \([0, 3]\). \( \square \)

(c) \( f(x) = \frac{1}{x} \) on \([\frac{1}{2}, \infty) \).

**Proof.** Let \( \varepsilon > 0 \), and fix \( \delta = \frac{\varepsilon}{4} \). If \( x, y \in \left[\frac{1}{2}, \infty\right) \) and \( |x - y| < \delta \), then \( \frac{1}{xy} \leq 4 \) and

\[
|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y-x}{xy} \right| = \left| \frac{1}{xy} \right| \cdot |x - y| 
\leq 4|x - y| < 4 \left( \frac{\varepsilon}{4} \right) = \varepsilon.
\]

Hence \( f \) is uniformly continuous on \([\frac{1}{2}, \infty) \). \( \square \)

(20.11) Find the following limits.

(a) \( \lim_{x \to a} \frac{x^2 - a^2}{x-a} \).

**Solution.** \( \lim_{x \to a} \frac{x^2 - a^2}{x-a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x \to a} (x + a) = 2a. \)
(b) \( \lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b}, \quad b > 0 \).

Solution. \[
\lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b} = \lim_{x \to b} \frac{(\sqrt{x} - \sqrt{b})(\sqrt{x} + \sqrt{b})}{(x - b)(\sqrt{x} + \sqrt{b})} = \lim_{x \to b} \frac{x - b}{(x - b)(\sqrt{x} + \sqrt{b})} = \lim_{x \to b} \frac{1}{\sqrt{x} + \sqrt{b}} = \frac{1}{2\sqrt{b}}.
\]

(c) \( \lim_{x \to a} \frac{x^3 - a^3}{x - a} \). Hint: \( x^3 - a^3 = (x - a)(x^2 + ax + a^2) \).

Solution. \[
\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2)}{x - a} = \lim_{x \to a} (x^2 + ax + a^2) = 3a^2.
\]

(20.17) Show that if \( \lim_{x \to a^+} f_1(x) = \lim_{x \to a^+} f_3(x) = L \) and if \( f_1(x) \leq f_2(x) \leq f_3(x) \) for all \( x \) in some interval \((a, b)\), then \( \lim_{x \to a^+} f_2(x) = L \).

Proof. Let \((x_n), (y_n)\) and \((z_n)\) be sequences in \((a, b)\) such that \( \lim x_n = \lim y_n = \lim z_n = a \); by hypothesis, the sequences \((f_1(x_n))\) and \((f_3(y_n))\) both converge to \( L \). But since \( f_1(x_n) \leq f_2(z_n) \leq f_3(y_n) \) for all \( x_n, y_n, z_n \in (a, b) \), it follows from the Squeeze Theorem for sequences that \( \lim f(z_n) = L \), which is the desired result. \( \square \)